Journal of Algebra **236**, 355–370 (2001)
doi:10.1006/jabr.2000.8486, available online at http://www.idealibrary.com on IDEAL®

# Maximum Order of Finite Abelian Subgroups in the Outer Automorphism Group of a Rank $n$ Free Group

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*Communicated by Walter Feit*

Received February 22, 2000

Let $F_n$ be a free group of rank $n$. Denote by Out $F_n$ its outer automorphism group, that is, its automorphism group modulo its inner automorphism group. For arbitrary $n$, by considering group actions on finite connected graphs, we derived the maximum order of finite abelian subgroups in Out $F_n$. Moreover, it is shown that the subgroups reaching this maximum order can be determined up to isomorphisms. © 2001 Academic Press

*Key Words:* free group; outer automorphism.

## 1. INTRODUCTION

Denote by $F_n$ the free group of rank $n$ and by

$$\text{Out } F_n = \text{Aut } F_n / \text{Inn } F_n$$

its outer automorphism group, namely, the automorphism group modulo the inner automorphisms.

The study of finite subgroups in Out $F_n$ has a close relationship with the study of group actions on connected graphs. An abstract *graph* consists of vertices and edges. For a finite connected graph $\Gamma$, as a topological space, its fundamental group $\pi_1(\Gamma)$ must be a free group. We call the rank $n$ of $\pi_1(\Gamma)$ the graph's *rank*. Note that the Euler number $\chi = 1 - n$. The *valence* of a vertex $p$ is the number of edges connecting to $p$, in which the edges with both ends coinciding with $p$ will be counted twice. If there is an edge in $\Gamma$ connecting two vertices $x$ and $y$ we say that $x$ and $y$ are *adjacent*.



0021-8693/01 $35.00

An *automorphism* of a graph $\Gamma$ is a one–one self-correspondence of its vertices and edges, preserving the relations between the edges and their ends. Here a reversal of edges is also taken into consideration. Denote by $\mathrm{Aut}\ \Gamma$ the group which consists of all such automorphisms. For any element $g \in \mathrm{Aut}\ \Gamma$, it induces an action on the free homotopic classes of closed paths in $\Gamma$, and thus an outer automorphism

$$g_* \in \mathrm{Out}\ \pi_1(\Gamma) \cong \mathrm{Out}\ F_n$$

in which $n$ is the rank of $\Gamma$. In fact, we get a correspondence that sends any subgroup $G < \mathrm{Aut}\ \Gamma$ homomorphically to a subgroup $G_* < \mathrm{Out}\ F_n$. In this case, we say that $G_*$ *acts on* $\Gamma$ and $G$ *realizes* $G_*$. If, in addition, the correspondence is an isomorphism, then we call it an *effective realization*.

Culler [2] and Zimmermann [5] observed independently that every finite subgroup of $\mathrm{Out}\ F_n$ can be realized by subgroups of automorphisms of some rank $n$ graph. Moreover, we have the following lemma, for which a proof can be found in [4].

LEMMA 1.1. *For any finite subgroup $G < \mathrm{Out}\ F_n$, there exists a connected graph $\Gamma$ (with rank $n$ and no vertices of valence 1 or 2) and a subgroup $H < \mathrm{Aut}\ \Gamma$ that realizes $G$ effectively.*

Therefore, for analyzing finite subgroups in $\mathrm{Out}\ F_n$, we only need to analyze automorphism groups on specific graphs. From this, Wang and Zimmermann [4] proved that, for finite subgroups of $\mathrm{Out}\ F_n$, their maximum order is 2 when $n = 1$, 12 when $n = 2$, and $2^n n!$ when $n \geq 3$. In [1], we found a number theoretical way to describe the maximum order of finite *cyclic* subgroups of $\mathrm{Out}\ F_n$ and established its asymptotic growth-rate formula.

In this article, also by using the previous lemma, we will determine the maximum order of finite *abelian* subgroups in $\mathrm{Out}\ F_n$ and subgroups reaching the maximum order.

THEOREM 1.1 (Main Theorem). *Fix a natural number $n \neq 2$. For any finite abelian subgroup $G < \mathrm{Out}\ F_n$, $|G| \leq 2^n$.*

*Every subgroup of $\mathrm{Out}\ F_n$ that reaches the maximum order is isomorphic to a direct product of copies of $\mathbf{Z}_2$ and $\mathbf{Z}_4$. (Here $\mathbf{Z}_k$ is the cyclic group with order $k$.) Moreover, every order-$2^n$ direct product of copies of $\mathbf{Z}_2$ and $\mathbf{Z}_4$ can be embedded into $\mathrm{Out}\ F_n$ isomorphically.*

*For $n = 2$, the maximum order of finite abelian subgroups of $\mathrm{Out}\ F_2$ is 6. Clearly any abelian group with order 6 is isomorphic to $\mathbf{Z}_6$.*

Many techniques used for studying group actions on graphs resemble those for studying finite symmetric groups. For any $n$-tuple $X$, its *symmet-*

*ric group* $S_X$ is the group of all permutations on it. If $X = \{1, 2, \dots, n\}$, denote it by $S_n$. The maximum orders of various categories of subgroups in $S_n$ have been discussed by many authors. In [3], for example, it is proved that the maximum order of its cyclic subgroups has the growth rate of $\exp(\sqrt{n \log n})$ with respect to $n$. We derived an accurate result on the maximum order of abelian subgroups in $S_n$ (Lemma 2.1), which also serves as the starting point for the induction for proving Theorem 1.1.

## 2. ABELIAN SUBGROUPS OF $S_n$ WITH MAXIMUM ORDER

For $S_1$ and $S_2$, obviously their abelian subgroups of maximum orders are just themselves, with orders 1 and 2 respectively. For $S_3$, it has only one abelian subgroup of maximum order; the cyclic subgroup generated by a 3-cycle, namely $\{(1), (123), (132)\}$, and the maximum order is 3. For $S_4$, the maximum order of its abelian subgroups is 4. There are exactly seven abelian subgroups with order 4. Here four of them, i.e., $\{(1), (12), (34), (12)(34)\}$, $\{(1), (13), (24), (13)(24)\}$, $\{(1), (14), (23), (14)(23)\}$, and $\{(1), (12)(34), (13)(24), (14)(23)\}$, are isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. The other three, namely $\{(1), (1234), (13)(24), (1432)\}$, $\{(1), (1243), (14)(23), (1342)\}$, and $\{(1), (1324), (12)(34), (1423)\}$, are isomorphic to $\mathbf{Z}_4$. It is interesting that these are the only patterns for abelian subgroups with maximum order.

LEMMA 2.1. *Given $n > 4$. Suppose that $G < S_n$ is an abelian subgroup with the maximum order. Then one and only one of the following three cases happens*:

1. $n \equiv 0 \mod 3$, *$G$ is generated by disjoint 3-cycles* $(\lambda_1, \lambda_2, \lambda_3), \dots, (\lambda_{n-2}, \lambda_{n-1}, \lambda_n)$.
2. $n \equiv 1 \mod 3$, *$G$ is generated by disjoint 3-cycles* $(\lambda_1, \lambda_2, \lambda_3), \dots, (\lambda_{n-6}, \lambda_{n-5}, \lambda_{n-4})$ *and an order 4 abelian group $G_0$ of permutations on* $\{\lambda_{n-3}, \lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$.
3. $n \equiv 2 \mod 3$, *$G$ is generated by disjoint 3-cycles* $(\lambda_1, \lambda_2, \lambda_3), \dots, (\lambda_{n-4}, \lambda_{n-3}, \lambda_{n-2})$ *and a flip* $(\lambda_{n-1}, \lambda_n)$.

*Let $c_n = n$ for $n \leq 4$. For $n \geq 4$, define $c_n = 3^k$ if $n = 3k$, $c_n = 4 \cdot 3^{k-1}$ if $n = 3k + 1$, and $c_n = 2 \cdot 3^k$ if $n = 3k + 2$ (here $k \in \mathbf{N}$). Then $|G| = c_n$.*

Note that our statements show that the abelian subgroups of $S_n$ reaching maximum order can be determined (almost unique) up to conjugations, and for any natural number $n$, $3^{(n-1)/3} \leq c_n \leq 3^{n/3}$.

*Proof.* Obviously the groups mentioned in the theorem are abelian, so we only need to prove that all the other abelian subgroups of $S_n$ are of orders less than $c_n$. Fix an abelian subgroup $G < S_n$ that acts on the $n$-tuple $\{1, 2, \dots, n\}$ and suppose that it realizes the maximum order.

For any $i$, $1 \le i \le n$, define

$$O_G(i) = \{j; 1 \le j \le n, \exists g \in G \text{ such that } g(i) = j\}$$

and call it *the orbit of G passing i*. If $i \ne j$, clearly $O_G(i)$ and $O_G(j)$ are either coincident or disjointed. Choose one element in each orbit set and denote them by $k_1, k_2, \dots, k_m$. Then

$$\{1, 2, \dots, n\} = O_G(k_1) \sqcup O_G(k_2) \sqcup \cdots \sqcup O_G(k_m),$$

in which $\sqcup$ stands for a disjoint union. Since $G$ is abelian, if $g, h \in G$ and $g(k_i) = h(k_i)$, $i = 1, 2, \dots, m$, then for any $j = f(k_i)$,

$$g(j) = g \circ f(k_i) = f \circ g(k_i) = f \circ h(k_i) = h \circ f(k_i) = h(j).$$

Therefore, $g \equiv h$, i.e., the elements in $G$ are determined by their actions on $k_1, k_2, \dots, k_m$. Thus

$$c_n \le |G| \le |O_G(k_1)|\,|O_G(k_2)| \cdots |O_G(k_m)|.$$

Now let natural numbers $x_1, x_2, \dots, x_m$ be given such that $x_1 + x_2 + \cdots + x_m = n$ and $x_1 x_2 \cdots x_m$ reaches the maximum value. We claim that there are only two possibilities, namely,

*Case* 1. One of these $x_i$'s equals 4 and the others equal 3.

*Case* 2. No more than two of these $x_i$'s equal 2 and all of the others equal 3.

Consequently $x_1 x_2 \cdots x_m = c_n$. In fact, when $x \ge 5$, then $x < 3(x-3)$, which implies that none of the $x_i$'s will be greater than or equal to 5. Clearly, no $x_i$ equals 1. $4 + 4 = 3 + 3 + 2$, $4 \cdot 4 < 3 \cdot 3 \cdot 2$, $4 + 2 = 3 + 3$, $4 \cdot 2 < 3 \cdot 3$, so if some $x_j = 4$, then all the other $x_i$'s are 3. $2 + 2 + 2 = 3 + 3$, $2 \cdot 2 \cdot 2 < 3 \cdot 3$, so if all the $x_i$'s are less than or equal to 3, then there are no more than two 2s. Thus, we have proved the claim.

Particularly, this claim implies that $|G| = c_n$ and $\{|O_G(k_i)|; 1 \le i \le m\}$ is such a set of $x_i$'s. Since these orbits are all invariant sets of $G$ and $G$ is maximal, we know that $G$ is the direct sum of its restrictions to them, which proves the assertions on the formats of $G$ in our theorem. ∎

## 3. FINITE ABELIAN SUBGROUPS OF Out $F_n$ WITH MAXIMUM ORDER

Let $F_n$ be the free group of rank $n$ generated by basis elements $a_1, a_2, \dots, a_n$. Let us first find out the abelian subgroups of Out $F_n$ which reach the order shown in our main theorem.

For each integer $i$, $1 \le i \le n$, let $g_i$ be automorphism on $F_n$ such that $g_i(a_j) = a_j$ whenever $j \ne i$ and $g_i(a_i) = a_i^{-1}$. These $g_i$'s commute with each other, since their actions on the generators of $F_n$ commute with each other. Let $F$ be the group of automorphisms on $F_n$ generated by $g_1, g_2, \dots, g_n$. Then $F$ is an abelian group of order $2^n$. Moreover, the only inner automorphism in $F$ is the trivial one. In fact, if $g \in F$ was a non-trivial inner automorphism, then there exists a non-trivial reduced word $w \in F_n$ such that $g(x) \equiv w^{-1}xw$ for all $x \in F_n$. This cannot happen when $n = 1$. If $n > 1$, suppose that $w = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots$ in which $\epsilon_k = \pm 1$. Then for any $j \ne i_1$, $w^{-1}a_j w$ is a reduced word of length at least 3, while $g(a_j) = a_j^{\epsilon}$, $\epsilon = \pm 1$, a contradiction. Therefore, $F$ is isomorphic to its quotient by the inner automorphism group, which is a subgroup of Out $F_n$ with order $2^n$. Denote the quotient by $F'$. Note that $F' \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$ ($n$ copies).

Instead of $\mathbf{Z}_2 \times \mathbf{Z}_2$, there is also a cyclic subgroup of order 4 in Out $F_2$. Consider the element $g \in$ Aut $F_n$ which sends $a_1$ to $a_2$ and $a_2$ to $a_1^{-1}$. The subgroup generated by it modulo the inner automorphism group gives such an example.

Let arbitrary non-negative integers $m_1$ and $m_2$ satisfying $n = m_1 + 2m_2$ be given. By similar discussions we know that there is a subgroup in Out $F_n$ which is isomorphic to the direct product of $m_1$ copies of $\mathbf{Z}_2$ and $m_2$ copies of $\mathbf{Z}_4$. Its order is $2^n$.

From Lemma 1.1, we know that every finite subgroup of Out $F_n$ can act effectively on some finite connected graph with rank $n$ and no vertex of valence 1 or 2. Therefore, Theorem 1.1 is implied by the following theorem.

THEOREM 3.1. *Fix a natural number $n \ne 2$. Suppose that $\Gamma$ is a connected graph, $\pi_1(\Gamma) \cong F_n$, $\Gamma$ has no vertex of valence* 1, *while at least one vertex of it has valence* $> 2$. *If $G$ is an abelian subgroup of* Aut $\Gamma$, *then $|G| \le 2^n$. Moreover, when $|G| = 2^n$, $G$ must be isomorphic to the direct product of copies of $\mathbf{Z}_2$ and $\mathbf{Z}_4$. For $n = 2$, the maximum order is* 6 *and the subgroups that reach this maximum order are isomorphic to $\mathbf{Z}_6$.*

We will prove the above theorem by reduction to absurdity, and consequently obtain Theorem 1.1. In the following paragraphs, we will first show that automorphisms on such a maximizing graph $\Gamma$ will always be determined by their induced permutations on the vertices, and then by using an induction on $n$ to show the results.

Obviously it is true for $n = 1$. Consider the case when $n = 2$. For the graph $\Gamma^*$ which consists of two vertices and three edges joining them, there are two elements $f_1, f_2 \in \mathrm{Aut}\ \Gamma^*$ defined as follows: $f_1$ permutes the edges while fixing the vertices and $f_2$ switches the vertices while keeping the edges invariant. Then $f_1$ commutes with $f_2$, and their orders are 3 and 2 respectively. So $f_1 \circ f_2 \in \mathrm{Aut}\ \Gamma^*$ has order 6. Direct calculations show that this is the maximum order among all choices of $\Gamma$ and $G$.

*Remark.* The above $\Gamma^*$ is in fact the only finite connected graph with no valence 1 or 2 vertex that accepts an order-6 group of automorphisms. Therefore, if $\Gamma$ is any rank-2 connected graph having an order-6 subgroup $G < \mathrm{Aut}\ \Gamma$, then after taking away inductively all the edges with one valence-1 end, and merging the edges joining at valence-2 vertices, we always get $\Gamma^*$. Note that $G$ has no fixed point in $\Gamma$. On the other hand, if $\Gamma$ is a rank-2 finite connected graph without a valence-1 vertex, and $G < \mathrm{Aut}\ \Gamma$ has a fixed point (not necessarily a vertex), then $|G| \leq 4$.

Suppose that the free group $F_2$ is generated by $a_1$ and $a_2$. The element in Out $F_2$ corresponding to the previous $(\Gamma^*, f_1 \circ f_2)$ is the quotient by Inn $F_2$ of the following $g \in \mathrm{Aut}\ F_2$:

$$g(a_1) = a_2^{-1}, \ g(a_2) = a_2 a_1.$$

For this $g$, if $g^2(x) \equiv w^{-1}xw$ in which $w$ is a reduced word with length $k$, then either $g^2(a_1) = a_1^{-1}a_2^{-1}$ or $g^2(a_2) = a_2 a_1 a_2^{-1}$ will have length at least $2k + 1$, and consequently $k \leq 1$. By enumeration, we see that such a word cannot exist. Therefore $g^2 \notin \mathrm{Inn}\ F_2$. Similarly, $g^3 \notin \mathrm{Inn}\ F_2$, while $g^6(x) \equiv w^{-1}xw \in \mathrm{Inn}\ F_2$ in which $w = a_1 a_2 a_1^{-1} a_2^{-1}$. Hence $g$ has order 6 in Out $F_2$.

Before we discuss the case when $n \geq 3$, let us first recall some special terms that will be used in the following discussions.

DEFINITION 3.1. An edge in a graph that has both ends coinciding with each other is called a *loop* (Fig. 1a). If for two vertices in a graph the total number of edges connecting them is greater than 1, then we call the

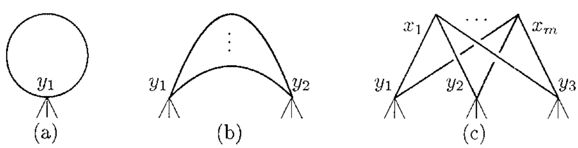


FIG. 1. (a) Loop, (b) multiple edges, and (c) $m$-star.

aggregate of these edges a *set of multiple edges* (Fig. 1b). A connected graph without loops, multiple edges, or vertices of valences 1 or 2 is called a *simple* graph.

For notational convenience, if a subgraph $\Gamma_0$ of a graph consists of three vertices $y_1, y_2, y_3$, $m(>1)$ valence-3 vertices $x_1, x_2, \dots, x_m$, and $3m$ edges $e_{ij}$ $(1 \le i \le m,\ 1 \le j \le 3)$ such that for each pair of $(i, j)$ the ends of $e_{ij}$ are $x_i$ and $y_j$; then we call $\Gamma_0$ an *m-star* (Fig. 1c).

DEFINITION 3.2. By *chain of valence-2 vertices* in $\Gamma$ we mean a set of valence-2 vertices $x_1, x_2, \dots, x_k \in \Gamma$ such that, for each $i < k$, there is an edge in $\Gamma$ connecting $x_i$ and $x_{i+1}$. If there is also an edge in $\Gamma$ connecting $x_1$ and $x_k$, namely, closing this chain, then we call it a *closed chain*.

If $X$ is a set of vertices in the graph $\Gamma$ that cuts $\Gamma$ into two parts (not necessarily connected), and by gluing $X$ back to each part, we obtain connected subgraphs of $\Gamma$ with positive rank, then we call $X$ a *separating set*. If $X = \{x\}$, $x$ is called a *separating point*, and if $|X| = 2$, $X$ is called a *separating pair*.

DEFINITION 3.3. Suppose that $X, Y \subseteq \Gamma$ are two subsets of graph $\Gamma$, and $G < \operatorname{Aut} \Gamma$ is a group of automorphisms. If $\forall f, g \in G$, $f|X = g|X$ implies that $f|Y = g|Y$, then we say that *the actions of $G$ on $Y$ are determined by the actions of $G$ on $X$*.

Now suppose contrarily that Theorem 3.1 is not true. Then there is a smallest natural number $n$ that invalidates the statement. Clearly $n > 2$. For this $n$, one can find a finite connected graph $\Gamma$ and an abelian subgroup $G < \operatorname{Aut} \Gamma$, such that $\pi_1(\Gamma) \cong F_n$, $\Gamma$ has no vertex of valence 1 while at least one vertex of it has valence $> 2$, and $|G| \ge 2^n$. If $|G| = 2^n$, one can also require that $G$ is not the direct sum of copies of $\mathbf{Z}_2$ and $\mathbf{Z}_4$. From now on, we fix this triple $(n, \Gamma, G)$.

Because $\Gamma$ is connected and has at least one vertex of valence $> 2$, we see that $\Gamma$ cannot have closed chains of valence-2 vertices. For a non-closed chain, since the action of $G$ on it will be determined by its action on the chain as a whole segment, we can simply merge them together to make a single edge and choose the resulting graph as $\Gamma$. $G$ can also be embedded into the automorphism group of this graph. Thus, we require without loss of generality that $\Gamma$ has no vertex of valence 2.

LEMMA 3.1. *Suppose that there is a triple $(n, \Gamma, G)$ satisfying the following requirements*: $n \in \mathbf{N}$; $n > 2$; $\Gamma$ *is a finite connected graph with no valence-*1 *vertex and at least one vertex of valence* $> 2$; *and* $G < \operatorname{Aut} \Gamma$ *is an abelian subgroup with order* $|G| \ge 2^n$. *If* $|G| = 2^n$, *we require that $G$ is not the direct sum of copies of* $\mathbf{Z}_2$ *and* $\mathbf{Z}_4$. *Choose such a triple that minimizes $n$ and satisfies that* $\Gamma$ *has no vertex of valence* 2 (*by the previous discussions, this can always be achieved*). *Then* $\Gamma$ *must be a simple graph*.

*Proof.* This is contained in the corollaries of the following two lemmas. ∎

LEMMA 3.2. *For $(n, \Gamma, G)$ in Lemma 3.1, $\Gamma$ has no separating point.*

*Proof.* Suppose contrarily that $\Gamma$ has separating points. Consider the separating point set $X = \{x\}$ and $Y$, a group of components in $\Gamma \setminus X$ such that the two subgraphs $Y \sqcup X$ and $\Gamma \setminus Y$ both have positive rank. We choose $X$ and $Y$ that minimize the number of vertices in $Y$ among all such choices. Then clearly $Y$ must be a connected component of $\Gamma \setminus X$.

Now let $\Gamma_{\mathrm{id}} = Y \sqcup X$, $\Gamma_g = g(\Gamma_{\mathrm{id}})$, $\forall g \in G$, and let $n_0$ be their common rank, $0 < n_0 < n$. Here id is the identity action on $\Gamma$. It is not difficult to prove that $\forall g, h \in G$, if $\Gamma_g \neq \Gamma_h$, then they intersect at most at $g(x)$. Choose a minimal set of elements $g_1 = \mathrm{id}, g_2, \ldots, g_k \in G$, such that

$$\{\Gamma_g;\ g \in G\} = \{\Gamma_{g_1}, \Gamma_{g_2}, \ldots, \Gamma_{g_k}\}.$$

In the graph $\Gamma \setminus \cup \{g(Y);\ g \in G\}$, let us remove inductively all the vertices of valence 1 and the edges connected to them. Denote the resulting graph by $\Gamma'$ and its rank by $n'$. Then $\Gamma'$ is an invariant set of $G$, and $G$ induces a group of automorphisms $G' < \mathrm{Aut}\,\Gamma'$. Since $n = n' + kn_0$, we know that $n' < n$. Moreover, the edges we removed first are exactly paths to certain vertices in $O_G(x)$. So the restricted actions of $G$ on them are completely determined by the induced actions of $G$ on $\Gamma'$ and every $\Gamma_g$. Define

$$G_1 = \{g \in G;\ g \text{ restricts to the identity on } \Gamma'\}.$$

Then $G' \cong G/G_1$, and all elements in $G_1$ are determined by their actions on $\cup\{\Gamma_g;\ g \in G\}$.

If, in addition, $\Gamma'$ has at least one vertex of valence $> 2$ and $n' \neq 2$, then $|G'| \leq 2^{n'}$. If this is the case or $n' = 2$ while $|G'| \leq 4$, choose $h_1 = \mathrm{id}$, $h_2, \ldots, h_m \in G_1$ such that $\{\Gamma_g;\ g \in G_1\} = \{\Gamma_{h_1}, \Gamma_{h_2}, \ldots, \Gamma_{h_m}\}$. Then obviously $m \leq k$. Put

$$G_2 = \{g \in G_1;\ \Gamma_{\mathrm{id}} \text{ is an invariant set of } g\}.$$

For any $h \in G_1$, there is an $i$, $1 \leq i \leq m$, such that $\Gamma_h = \Gamma_{h_i}$. Consequently $h_i h^{-1}$ keeps $\Gamma_{\mathrm{id}}$ invariant. Therefore, $|G_1 : G_2| \leq m$. So $|G : G_2| = |G'|\,|G_1 : G_2| \leq m2^{n'}$.

If $n' = 2$ and $|G'| > 4$, then by the remark following Theorem 3.1, $G'$ is a cyclic group of order 6 and $G'$ has no fixed point in $\Gamma \setminus \bigcup\{\Gamma_g;\ g \in G\}$. Consequently, $O_G(x) = \{g(x);\ g \in G\}$, the orbit passing the separating point, has at least two points. Define $G_1$, $G_2$, and $h_1 = \mathrm{id}, h_2, \ldots, h_m \in G_1$

similarly, then $2m \leq k$, while $|G:G_1| = 6$, $|G_1:G_2| = m$. Thus $|G:G_2| \leq 6 \cdot k/2 < k2^2$.

If $\Gamma'$ is a single point, then $G_1 \cong G$. Define $G_2$ similarly; we see that $|G:G_2| = |G_1:G_2| \leq m = m2^{n'}$.

If all the vertices in $\Gamma'$ have valence 2, then $\Gamma'$ must be a closed chain of valence-2 vertices, and $n' = 1$. The action of any element $g \in G$ on it is determined by the image of a single point and whether $g$ preserves the direction of the chain. Choose this point to be one that has a path outside $\Gamma'$ connecting to $\Gamma_{\mathrm{id}}$, then its image under the action of $G$ is determined by the actions of $G$ on $\Gamma_{\mathrm{id}}$. Define

$$G_2 = \{g \in G;\, \Gamma_{\mathrm{id}} \text{ is an invariant set of } g, g \,|\, \Gamma' = \mathrm{id}_{\Gamma'}\},$$

then similarly $|G:G_2| \leq 2k$.

In all the cases, $|G:G_2| \leq k2^{n'}$ and $G_2$ induces a group of automorphisms $G_2'$ on $\Gamma_{\mathrm{id}}$. The rank of $\Gamma_{\mathrm{id}}$ is $n_0 < n$, and $G_2'$ fixes at least the separating point $x \in \Gamma_{\mathrm{id}}$, so that when $\Gamma_{\mathrm{id}}$ is not a closed chain of valence-2 vertices, $|G_2'| \leq 2^{n_0}$. When it is such a closed chain, because $G_2'$ fixes the separating point $x \in \Gamma_{\mathrm{id}}$, we still have $|G_2'| \leq 2 = 2^{n_0}$.

Since $G$ is commutative, $\forall h \in G_2$ and $\forall i$, $1 \leq i \leq m$, we have $h \circ g_j \,|\, \Gamma_{\mathrm{id}} = g_j \circ h \,|\, \Gamma_{\mathrm{id}}$. This implies that the action of $h$ on $\Gamma$ is uniquely determined by its action on $\Gamma_{\mathrm{id}}$ or, equivalently, $G_2 \cong G_2'$.

To sum up, the order of $G$ is

$$2^n \leq |G| = |G:G_2|\,|G_2'| \leq k2^{n'}2^{n_0}.$$

However, $2^{n_0} \geq 2$, so $k2^{n_0} \leq 2^{kn_0}$, which implies that $|G| \leq 2^{n'+kn_0} = 2^n$. So the above inequalities are in fact equalities, and $|G| = 2^n$. Clearly, $k2^{n_0} = 2^{kn_0}$ implies that either $k = 2$, $n_0 = 1$, or $k = 1$.

If $k = 1$, then $\Gamma_{\mathrm{id}}$ itself is also an invariant set of $G$, so $G$ is the direct sum of $G'$ and $G_2'$, $|G'| = 2^{n'}$, $|G_2'| = 2^{n_0}$, and $|G| = 2^n$. Then, by the assumptions we made previously, $G'$ and $G_2'$ are isomorphic to direct products of copies of $\mathbf{Z}_2$ and $\mathbf{Z}_4$, and so is $G$, which cannot happen.

If $k = 2$ and $n_0 = 1$, then $\{g(\Gamma_{\mathrm{id}});\, g \in G\} = \{\Gamma_{\mathrm{id}}, \Gamma_{g_2}\}$, and $\Gamma_{\mathrm{id}} \cup \Gamma_{g_2}$ is an invariant set of $G$. If the separating point $x$ is invariant, then $|G'| \leq 2^{n-2}$, $|G_1| \leq 4$, and $G$ is isomorphic to their direct product. If $x$ is not stable, then again $|G'| = 2^{n-2}$. However, $G_1$ keeps the vertices in $O_G(x)$ invariant, so $|G_1| \leq 2$ and $|G| < 2^n$. This also contradicts our assumptions on $G$.

Therefore, $\Gamma$ does not contain separating points. ∎

If $n > 1$, the end of any loop will be a separating point. Thus, we have the following.

COROLLARY 3.1. *For $(n, \Gamma, G)$ in Lemma* 3.1, $\Gamma$ *has no loop.*

LEMMA 3.3. *For $(n, \Gamma, G)$ in Lemma* 3.1, $\Gamma$ *has no separating pair.*

*Proof.* From Lemma 3.2, we know that $\Gamma$ has no separating point. Suppose contrarily that $\Gamma$ has separating pairs. Consider the separating set $X = \{x_1, x_2\}$ and a group of components $Y$ of $\Gamma \backslash X$ such that the two subgraphs $Y \cup X$ and $\Gamma \backslash Y$ both have positive ranks. Again we choose $X$ and $Y$ such that the number of vertices in $Y$ is minimized.

Now let $\Gamma_{\mathrm{id}} = Y \cup X$, $\Gamma_g = g(\Gamma_{\mathrm{id}})$, $\forall g \in G$, and let $n_0$ be their common rank, $0 < n_0 < n$. $\forall g \in G$, if $\Gamma_g \neq \Gamma_{\mathrm{id}}$, then it is not difficult to prove that neither $g(x_1)$ nor $g(x_2)$ can be in $Y$. Therefore, $\Gamma_g$ and $\Gamma_{\mathrm{id}}$ intersect at most in $X$. Consequently, for any $g, h \in G$, if $\Gamma_g \neq \Gamma_h$ they also intersect at most in $g(X)$. Choose a minimal set of elements $g_1 = \mathrm{id}, g_2, \ldots, g_k \in G$, such that

$$\{\Gamma_g;\, g \in G\} = \{\Gamma_{g_1}, \Gamma_{g_2}, \ldots, \Gamma_{g_k}\}.$$

Consider the graph obtained from $\Gamma$ by replacing each $\Gamma_g$ by an edge from $g(x_1)$ to $g(x_2)$, $\forall g \in G$. Remove inductively all the vertices of valence 1 and the edges connected to them. Denote the resulting graph by $\Gamma'$ and its rank by $n'$. Then $G$ induces a group of automorphisms $G'$ on $\Gamma'$. Since $n = n' + kn_0$, we know that $n' < n$.

If $\Gamma'$ has at least one vertex of valence $> 2$, or $\Gamma'$ is actually a single point, define

$$G_1 = \{g \in G;\, g \text{ corresponds to the identity action on } \Gamma'\},$$

$$G_2 = \{g \in G_1;\, \Gamma_{\mathrm{id}} \text{ is an invariant set of } g\}.$$

If $\Gamma'$ is a loop of valence 2 vertices, define

$$G_2 = \{g \in G;\, \Gamma_{\mathrm{id}} \text{ is an invariant set of } g, g \,|\, (X \cup \Gamma') = \mathrm{id}_{X \cup \Gamma'}\}.$$

Similar to the proof of Lemma 3.2, we can prove that $|G : G_2| \leq k2^{n'}$ if $n' \neq 2$ or $n' = 2$ while $|G'| \leq 4$. If $n' = 2$, $G' \cong \mathbf{Z}_6$, then by the remark following Theorem 3.1 the separating set cannot be an invariant set. Otherwise the generator of $G'$ fixes at least one point in the edge connecting $x_1$ and $x_2$, which is impossible. Recall that in the construction of $\Gamma'$, we have defined such an edge to substitute the component $\Gamma_{\mathrm{id}}$ being cut off from $\Gamma$. Consequently $|G : G_2| \leq 6 \cdot k/2 < k2^{n'}$.

Because $G_2$ fixes $x_1, x_2 \in \Gamma_{\mathrm{id}}$, $|G_2| \leq 2^{n_0}$. Therefore $|G| \leq k2^{n'+n_0} \leq 2^n$. However, we have assumed that $|G| \geq 2^n$ at the beginning of the proof, so the equality must hold. This implies that $|G'| \leq 4$ when $n' = 2$, and either $k = 2$, $n_0 = 1$, or $k = 1$.

If $k = 1$, then the separating set $\{x_1, x_2\}$ is invariant under $G$, although some elements in $G$ may switch $x_1$ and $x_2$. Similar to the proof of Lemma 3.2, we know that $G$ is isomorphic to a subgroup of the direct product of $G \mid \Gamma'$ and $G \mid \Gamma_{\mathrm{id}}$. Then the equality on orders of these groups forces $G$ to be exactly the direct product. If $k = 2$, $n_0 = 1$, and $G$ keeps $\{x_1, x_2\}$ invariant, then $G$ is isomorphic to the direct product of $G \mid \Gamma'$ and $G \mid \bigcup\{\Gamma_h; h \in G\}$. Obviously the $\Gamma'$ we defined cannot be a single point, so the rank of $\Gamma'$ is again positive. In both cases, we see that $G$ is isomorphic to a direct product of copies of $\mathbf{Z}_2$ and $\mathbf{Z}_4$. If $k = 2$ and the set $\{x_1, x_2\}$ is not invariant, then similar to the previous discussions it can be showed that $|G| < 2^n$, which is also a contradiction.

To sum up, the separating pairs cannot exist. ∎

COROLLARY 3.2. *For $(n, \Gamma, G)$ in Lemma 3.1, $\Gamma$ has no multiple edges.*

*Proof.* Suppose contrarily that $\Gamma$ has exactly $m(> 1)$ multiple edges connecting two vertices $x_1$ and $x_2$. By Lemma 3.3, $\{x_1, x_2\}$ cannot be a separating pair, which implies that the rank of $\Gamma$ is $m - 1$ and $m \le 3$. Denote the numbers of vertices and edges in $\Gamma$ by $v$ and $e$ respectively, then $e \ge 3v/2$ and $\chi(\Gamma) = 1 - (m - 1) = v - e \le -v/2$, which implies $v \le 2(m - 2)$. Since $v \ge 2$, we know that $m = 3$, $v = 2$, and $e = 3$. $\Gamma$ consists of exactly two points $x_1, x_2$ and three edges connecting $x_1$ and $x_2$, and its rank is 2. This has already been discussed in the remark following Theorem 3.1. ∎

LEMMA 3.4. *For $(n, \Gamma, G)$ in Lemma 3.1, $n > 18$. Namely, for $n \le 18$, Theorem 3.1 holds.*

*Proof.* By the previous two corollaries, we know that under the assumptions we have made on $\Gamma$ it has no loop or multiple edges, and all vertices are of valence $> 2$. Therefore, any automorphism on $\Gamma$ is completely determined by its action on the vertices, which is a permutation.

Denote by $v_k$ the number of valence-$k$ vertices in $\Gamma$. Then $\Gamma$ has a total $v = v_3 + v_4 + \cdots$ vertices and $e = (3v_3 + 4v_4 + \cdots)/2$ edges. But the Euler characteristic $\chi(\Gamma) = 1 - n = v - e$, so

$$2(n - 1) = v_3 + 2v_4 + 3v_5 + \cdots \ge v.$$

By Lemma 2.1, we know that $|G| \le c_v \le c_{2(n-1)}$. Direct calculations show that for $n \le 18$, $c_{2(n-1)} < 2^n$, and thus the lemma is proved. ∎

LEMMA 3.5. *For $(n, \Gamma, G)$ in Lemma 3.1, $\Gamma$ does not contain any $m$-star if $m > 1$.*

*Proof.* By the previous discussions, we may assume that $\Gamma$ has no loop or multiple edges and $n > 18$. Suppose contrarily that the lemma is not true. Then, for some triple $y_1 \neq y_2 \neq y_3$, there are $m$ ($> 1$) valence-3 vertices in $\Gamma$ that are simultaneously adjacent to them all. Choose a triple that maximizes $m$, and denote these $m$ vertices by $x_1, x_2, \ldots, x_m$ (see Fig. 1).

We claim that $\forall g \in G$ and $x_i$, $g(x_i)$ cannot be some $y_j$. In fact, if, say, $g(x_1) = y_1$, then $y_1$ is a vertex of valence 3, which implies $m = 2$ or 3 since all the $x_i$'s are adjacent to it.

If $m = 3$, clearly $g(\{y_1, y_2, y_3\}) = \{x_1, x_2, x_3\}$. Thus there cannot be any other edges connecting one of the $x_i$'s or $y_j$'s with the rest of $\Gamma$. Since $\Gamma$ is connected, this implies that $\Gamma$ consists of exactly the $x_i$'s and $y_j$'s and all of the edges connecting them. Thus, the rank of $\Gamma$ is $4 \leq 18$, in contradiction to Lemma 3.4. If $m = 3$, suppose that $y_1$ is adjacent to $x_1, x_2, z$. Since $g(x_2)$ is also adjacent to them, $g(x_2)$ must be $y_2$ or $y_3$, say $y_2$. The subgraph of $\Gamma$ that consists of the $x_i$'s, the $y_j$'s, $z$, and all of the edges connecting them has rank $3 < n - 1$. So $\{y_3, z\}$ is a separating pair, in contradiction to Lemma 3.3. Therefore, the case in our claim is the only possibility left.

Now let $\Gamma_{\mathrm{id}}$ be the subgraph of $\Gamma$ which consists of the $x_i$'s and $y_j$'s and all of the edges starting from some $x_i$. Put $\Gamma_g = g(\Gamma_{\mathrm{id}})$, $\forall g \in G$. Then the previous claim implies that for any $g, h \in G$, if $\Gamma_g \neq \Gamma_h$, they intersect at most at $\{g(y_1), g(y_2), g(y_3)\}$. Choose a minimal set of elements $g_1 = \mathrm{id}$, $g_2, \ldots, g_k \in G$, such that

$$\{\Gamma_g;\, g \in G\} = \{\Gamma_{g_1}, \Gamma_{g_2}, \ldots, \Gamma_{g_k}\}.$$

Consider the graph obtained from $\Gamma$ by replacing each $\Gamma_g$ with a subgraph consisting of a valence-3 vertex and three edges connecting it with the $g(y_j)$'s. Remove from it inductively all the vertices of valence 1 and the edges connected to them. Denote the resulting graph by $\Gamma'$ and its rank by $n'$. Then $G$ induces a group of automorphisms $G'$ on $\Gamma'$. Since $n = n' + 2k(m - 1)$, we know that $n' < n$.

Now define

$$G_1 = \{g \in G;\, g \text{ corresponds to the identity action on } \Gamma'\},$$

$$G_2 = \{g \in G_1;\, \Gamma_{\mathrm{id}} \text{ is an invariant set of } g\}.$$

Similar to the proof of Lemma 3.2, we can prove that $|G : G_2| \leq k2^{n'}$ if $n' \neq 2$ or $n' = 2$ while $|G'| \leq 4$. If $n' = 2$, $G' \cong \mathbf{Z}_6$, consider the minimum subset of $\{y_1, y_2, y_3\}$ that constitutes a separating set. Then, by the remark following Theorem 3.1, the separating set cannot be an invariant set, otherwise the generator of $G'$ fixes at least the vertex adjacent to $y_1, y_2, y_3$ all, which is impossible. Remember that in the construction of $\Gamma'$ we have defined such a vertex to substitute for the $Y$ being cut off from $\Gamma$. Consequently $|G_1 : G_2| \leq k/2$ and $|G : G_2| \leq 6 \cdot k/2 < k2^{n'}$.

$|G_2|$ is not greater than the maximum order of the finite abelian group of automorphisms on $\Gamma_{\text{id}}$. No $g \in G$ can send an $x_i$ to some $y_i$. Hence any automorphism on $\Gamma_{\text{id}}$ obtained by restricting some $g \in G$ to it is determined by the induced permutation on the valence-3 vertices $x_1, x_2, \dots, x_m$ and on the other three vertices $y_1, y_2, y_3$. By Lemma 2.1, $|G_2| \leq 3c_m < 2^{2(m-1)}$,

$$|G| = |G : G_1|\,|G_1 : G_2|\,|G_2| < k2^{n' + 2(m-1)} \leq 2^n,$$

a contradiction.

Hence this kind of subgraph in $\Gamma$ cannot exist, either. ▮

## 4. PROOF OF THEOREM 1.1 AND THEOREM 3.1

Now let us prove our Theorem 3.1 and consequently the main result. As in the previous section, choose the triple $(n, \Gamma, G)$ as in Lemma 3.1. Then $\Gamma$ must be a simple graph without $m$-star if $m > 1$ and $n \geq 18$.

*Notation.* Let $\Gamma^{(0)}$ be the set of all vertices in $\Gamma$. For any subset $X \subseteq \Gamma^{(0)}$, denote by $x_k$ the number of valence $k$ vertices in $X$, and define $\|X\| = x_3 + 2x_4 + 3x_5 + \cdots$. Clearly $|X| \leq \|X\|$, $\|X \sqcup Y\| = \|X\| + \|Y\|$. In addition, from the proof of Lemma 3.4, we know that

$$\|\Gamma^{(0)}\| \leq 2(n-1).$$

Fix $x_0 \in \Gamma^{(0)}$ and let $X_0 = \{x_0\}$. Let $G_0$ be the set of automorphisms in $G$ that fix $X_0$ invariant. Then clearly $G_0$ is a subgroup of $G$ and $|G : G_0| \leq |\Gamma^{(0)}| \leq 2(n-1)$. Now we will construct $X_k$ and $G_k$ inductively as follows.

LEMMA 4.1. *Whenever $X_{k-1} \neq \Gamma^{(0)}$, we can find the point set $X_k \subseteq \Gamma^{(0)}$ and the subgroup $G_k < G_{k-1}$ satisfying the following requirements: $X_k \supseteq X_{k-1}$; $X_k \neq X_{k-1}$; $|G_{k-1} : G_k| \leq C^{\|X_k \setminus X_{k-1}\|}$; and the restriction of $G_k$ to $X_k$ equals identity. Here the constant*

$$C = \left(2^{18}/18\right)^{1/35} = 1.315\dots.$$

The process stops when some $X_k = \Gamma^{(0)}$. Then $G_k$ keeps all the vertices in $\Gamma$ invariant. Since $\Gamma$ is a simple graph, we see that $G_k$ is a trivial group. Therefore,

$$\begin{aligned} |G| &= |G : G_0|\,|G_0 : G_1| \cdots |G_{k-1} : G_k| \\ &\leq 2(n-1)C^{\|X_1 \setminus X_0\| + \|X_2 \setminus X_1\| + \cdots + \|X_k \setminus X_{k-1}\|} \\ &\leq 2(n-1)C^{\|\Gamma^{(0)}\| - 1} \leq 2(n-1)C^{2n-3}. \end{aligned}$$

Consequently $2(n-1)C^{2n-3} \geq 2^n$. By taking the logarithm, we see that $f(n) \geq 0$, in which $f(x) = \log(2C^{-3}) + \log(x-1) + x\log(C^2/2)$. By Lemma 3.4, $n \geq 19$. However, because $C < \sqrt{2}/\exp(1/36)$, the derivative $f'(x) < 0$ when $x \geq 19$. Now $f(19) < 0$, so $\forall x \geq 19$, $f(x) < 0$. The statement of Theorem 3.1 follows from this contradiction.

*Proof of Lemma* 4.1. First, fix an $x \in \Gamma^{(0)} \setminus X_{k-1}$ such that in $\Gamma$ there is a $y \in X_{k-1}$ adjacent to $x$. Let

$$X'_k = O_{G_{k-1}}(x) = \{g(x);\, g \in G_{k-1}\}.$$

By definition, the elements in $G_{k-1}$ are automorphisms of $\Gamma$ that restrict to identities in $X_{k-1}$. Therefore, each $x' \in X'_k$ is adjacent to $y$ and has the same valence as that of $x$. $X'_k \cap X_{k-1} = \varnothing$.

If $x$ has valence $\geq 4$, put

$$X_k = X_{k-1} \cup X'_k$$

$$G_k = \{g \in G_{k-1};\, g \text{ restricts to the identity on } X_k\}.$$

Then $|X'_k| = |X_k \setminus X_{k-1}| \leq \|X_k \setminus X_{k-1}\|/2$, and $G_k$ is a subgroup of $G_{k-1}$. Moreover,

$$|G_k : G_{k-1}| \leq |X'_k| \leq 3^{|X'_k|/3} \leq 3^{\|X_k \setminus X_{k-1}\|/6} \leq C^{\|X_k \setminus X_{k-1}\|}.$$

If $x$ has valence 3 and $|X'_k| = 1$, then define $X_k$ and $G_k$ in the same way. $G_k = G_{k-1}$, so we still have $|G_k : G_{k-1}| = 1 \leq C^{\|X_k \setminus X_{k-1}\|} = C$.

In the following paragraph, we suppose that the valence of $x$ is 3, and $|X'_k| > 1$. Then all $x' \in X'_k$ have valence 3. By Lemma 3.5, the three points $y, y_1, y_2$ adjacent to $x$ cannot all be in $X_{k-1}$. We already know that $y \in X_{k-1}$. If $y_1 \in X_{k-1}$, $y_2 \in X'_k$, then $\{y, y_1\}$ is a separating pair that separates $x, y_2$ with the remaining parts of $\Gamma$, which is ruled out by Lemma 3.3 (Fig. 2a). If $y_1, y_2 \in X'_k$, then $y$ separates $X'_k$ with the remaining parts of $\Gamma$. By Lemma 3.3, we know that the graph obtained from $\Gamma$ by removing $X'_k$ and the edges connecting to them has rank 0.

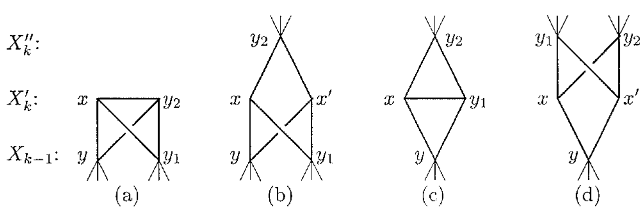


FIG. 2. Simple cases of Lemma 4.1.

Simple calculation shows that the rank $n$ of $\Gamma$ equals $|X'_k|$, and $G$ is determined by its action on $x$. But then $|G| \leq |X'_k| < 2^{|X'_k|} = 2^n$, which contradicts our assumption on $G$.

If $y_1 \in X_{k-1} \cup X'_k$, $y_2 \in \Gamma^{(0)} \setminus (X_{k-1} \cup X'_k)$, then $\forall x' \in X'_k$, and its adjacent vertices also satisfy this same distribution property. Let $X''_k$ be the set of vertices in $\Gamma^{(0)} \setminus (X_{k-1} \cup X'_k)$ that are adjacent to some points in $X'_k$, put

$$X_k = X_{k-1} \cup X'_k \cup X''_k,$$
$$G_k = \{g \in G_{k-1};\ g \text{ restricts to the identity on } X'_k\}.$$

Then $G_k \,|\, X_k = \mathrm{id}_{X_k}$ and $|G_k : G_{k-1}| = |X'_k|$. Moreover, $\forall x'' \in X''_k$, $\|x''\|$ is at least one third of the number of vertices in $X'_k$ adjacent to it. Thus $\|X_k \setminus X_{k-1}\| \geq \lceil 4|X'_k|/3 \rceil$, in which $\lceil\ \rceil$ means taking the least integer that is not less than the operand. Direct calculations show that this implies $|G_k : G_{k-1}| \leq C^{\|X_k \setminus X_{k-1}\|}$ whenever $|X'_k| \neq 3$ or $|X'_k| = 3$, $\|X''_k\| > 1$.

When $|X'_k| = 3$, $X''_k = \{y_2\}$, and $\|y_2\| = 1$, there are two subcases. If $y_1 \in X_{k-1}$, then the edges which start from $X'_k$ form at least a 2-star, which is already ruled out by Lemma 3.5 (Fig. 2b). If $y_1 \in X'_k$, then like $(x, y_1)$, the vertices in $X'_k$ can be divided into adjacent pairs (Fig. 2c). However, this implies that $|X'_k|$ is an even number, which is still impossible.

If $y_1, y_2 \in \Gamma^{(0)} \setminus (X_{k-1} \cup X'_k)$, then for all $x' \in X'_k$, their adjacent vertices also satisfy this same property. Again let $X''_k$ be the set of vertices in $\Gamma^{(0)} \setminus (X_{k-1} \cup X'_k)$ that are adjacent to some points in $X'_k$, and put

$$X_k = X_{k-1} \cup X'_k \cup X''_k,$$
$$G_k = \{g \in G_{k-1};\ g \text{ restricts to the identity on } X'_k, y_1, y_2\}.$$

Since $X'_k = O_{G_{k-1}}(x)$, $X''_k = O_{G_{k-1}}(\{y_1, y_2\})$. Because $G$ is abelian, we know that for any element in $G$ its action on the orbit passing $y_1$ or $y_2$ is determined by its action on $y_1$ or $y_2$. Particularly, $G_k \,|\, X''_k = \mathrm{id}_{X''_k}$. Thus, $G_k \,|\, X_k = \mathrm{id}_{X_k}$. We derive again that $\forall x'' \in X''_k$, $\|x''\|$ is at least one-third of the number of vertices in $X'_k$ adjacent to it. Hence,

$$\|X_k \setminus X_{k-1}\| = |X'_k| + \|X''_k\| \geq \lceil 5|X'_k|/3 \rceil.$$

If for any $g \in G_{k-1}$ which equals identity on $X'_k$, $g$ also fixes $y_1$ and $y_2$, then $|G_k : G_{k-1}| = |X'_k|$. This implies that $|G_k : G_{k-1}| \leq C^{\|X_k \setminus X_{k-1}\|}$ since $\|X''_k\| \geq 2$. Otherwise $|G_k : G_{k-1}| = 2|X'_k|$. In this case, direct computations show that $|G_k : G_{k-1}| \leq C^{\|X_k \setminus X_{k-1}\|}$ whenever $|X'_k| > 4$ or $|X'_k| \leq 4$, $\|X''_k\| \geq 4$.

What is left is only the case when $|X'_k| \leq 4$, $\|X''_k\| < 4$, and there is a $g \in G_{k-1}$ which fixes $X'_k$ while sending $y_1$ to $y_2$. In this case, if $y_1$ is adjacent to some vertices in $X_k'$, then $y_2$ is also adjacent to them. Simple reasoning shows that $\|X_k''\|$ is even, i.e., it cannot be 3. However, if $\|X_k''\| = 2$, we get an $|X_k'|$-star (Fig. 2d), in contradiction to Lemma 3.5. Hence, these cases are all impossible.

To sum up, the required $X_k$ and $G_k$ can be found whenever $X_{k-1} \neq \Gamma^{(0)}$, thus the lemma and our main results are proved. ∎

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